JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **163**, 86–103 (1992)

Mathematical Analysis of a Chemical Reaction with Lumped Temperature and Strong Absorption

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Submitted by Ivar Stakgold

Received August 6, 1990

We consider a mathematical model for the evolution of a single reactant and of the temperature in an isothermal catalyst. The temperature is assumed to be spatially homogeneous and the absorption term is supposed to be nonlipschitzian. Existence and uniqueness of a regular solution is proved together with some bounds. The steady-state problem is also investigated and information about the "dead-core" of the reactant are given.

1. INTRODUCTION

In this paper we consider a model for the evolution of a single reactant u and of the temperature v in an isothermal catalyst. We assume that the temperature is spatially homogeneous. The equations of the model are

$$\partial u/\partial t = \Delta u - \phi g(v) f(u) \qquad \text{in } \Omega \times (0, \infty) \tag{1.1a}$$

$$\alpha\, \partial u/\partial \nu = 1 - u \qquad \text{on } \partial\Omega \times (0, \infty) \tag{1.1b}$$

$$dv/dt = k(1 - v) + \lambda \phi g(v) \int_\Omega f(u)\, dx \qquad \text{in } (0, \infty) \tag{1.1c}$$

$$u(x, 0) = u_0(x) \geqslant 0 \qquad \text{in } \Omega \tag{1.1d}$$

$$v(0) = v_0 \geqslant 0. \tag{1.1e}$$

0022-247X/92 $3.00

Here, $f(u)=u^p$ if $u \geqslant 0$ and $f(u)=0$ if $u<0$; $g(v)=\exp(\gamma(1-1/v))$ if $v>0$ and $g(v)=0$ if $v \leqslant 0$.

The parameters k, ϕ, α, γ, p are positive while λ can be positive (*exothermic reaction*), negative (*endothermic reaction*), or zero (*isothermic reaction*).

This model was first proposed by Aris [2] as a first approximation of the nonisothermal model in which the temperature is spatially distributed.

The model (1.1) is also investigated in Vega [9], where $f(u)\, g(v)$ is replaced by a C^1 function $F(u, v)$ and sufficient conditions for the global asymptotic stability of the steady state are given.

When $p \geqslant 1$, the solution $u(x, t)$ is strictly positive at any positive time t. If $0<p<1$, as often happens in practice, then $-\phi g(v) f(u)$ is called a *strong absorption* term and $u(x, t)$ can be zero in a nonempty set $D(t)$. The region $D(t)$ is called the *dead-core* at time t. In $D(t)$ no reaction takes place and therefore it would be useful to avoid the existence of these regions.

The problem of the dead-core for a single parabolic equation with strong absorption has been studied in recent years by a number of authors. For a review of the subject see Stakgold [8].

The aim of this paper is to investigate qualitative properties of the system (1.1) using comparison techniques, to give sufficient conditions on the parameters for the existence and the nonexistence of the dead-core and to study some of its properties.

The rest of the paper is organized as follows.

In Section 2 we give the definition of lower and upper solutions for (1.1) and then we prove an *existence and uniqueness* result constructing a nonlinear iterative scheme similar to the one proposed by Diaz and Stakgold [4].

In Section 3 we study the *steady-state problem.* We give a uniqueness result for the endothermic case and a sufficient condition on γ for the uniqueness in the exothermic case. Then we analyze the stationary dead-core.

In Section 4 the time-dependent problem is analyzed. Bounds for the solution of (1.1) and some results about its asymptotic behavior are given. Moreover the dead-core $D(t)$ is studied.

2. Existence and Uniqueness

Let $(\underset{\sim}{u}, \underset{\sim}{v})$, $(\tilde{u}, \tilde{v})$ be two pairs of "smooth" functions (that is $\underset{\sim}{u}$, $\tilde{u}$ are in $C^{2,1}(\Omega \times (0, T)) \cap C^{1,0}(\bar{\Omega} \times [0, T])$ and $\underset{\sim}{v}$, $\tilde{v}$ are in $C^1((0, T)) \cap C^0([0, T])$, where T is any positive number. We say that $(\underset{\sim}{u}, \underset{\sim}{v})$, $(\tilde{u}, \tilde{v})$ are pairs of lower and upper solutions (l.u.s.) if

(I) For $\lambda < 0$:

$$\tilde{u}_t - \Delta\tilde{u} + \phi g(\underset{\sim}{v}) f(\tilde{u}) \geqslant 0 \qquad \text{in } \Omega \times (0, T) \tag{2.1a}$$

$$\tilde{v}_t - k(1 - \tilde{v}) - \lambda\phi g(\tilde{v}) \int_\Omega f(\underset{\sim}{u})\, dx \geqslant 0 \qquad \text{in } (0, T) \tag{2.1b}$$

$$\underset{\sim}{u}_t - \Delta\underset{\sim}{u} + \phi g(\tilde{v}) f(\underset{\sim}{u}) \leqslant 0 \qquad \text{in } \Omega \times (0, T) \tag{2.1c}$$

$$\underset{\sim}{v}_t - k(1 - \underset{\sim}{v}) - \lambda\phi g(\underset{\sim}{v}) \int_\Omega f(\tilde{u})\, dx \leqslant 0 \qquad \text{in } (0, T) \tag{2.1d}$$

$$\tilde{u} + \alpha\tilde{u}_\nu - 1 \geqslant 0 \qquad \text{on } \partial\Omega \times (0, T) \tag{2.1e}$$

$$\underset{\sim}{u} + \alpha\underset{\sim}{u}_\nu - 1 \leqslant 0 \qquad \text{on } \partial\Omega \times (0, T) \tag{2.1f}$$

$$\underset{\sim}{u}(x, 0) \leqslant u_0(x) \leqslant \tilde{u}(x, 0) \qquad \text{in } \Omega \tag{2.1g}$$

$$\underset{\sim}{v}(0) \leqslant v_0 \leqslant \tilde{v}(0) \tag{2.1h}$$

$$\underset{\sim}{u} \leqslant \tilde{u} \quad \text{in } \Omega \times (0, T) \qquad \underset{\sim}{v} \leqslant \tilde{v} \quad \text{in } (0, T) \tag{2.1i}$$

(II) For $\lambda > 0$ (2.1b) and (2.1d) are replaced by

$$\tilde{v}_t - k(1 - \tilde{v}) - \lambda\phi g(\tilde{v}) \int_\Omega f(\tilde{u})\, dx \geqslant 0 \qquad \text{in } (0, T) \tag{2.1b$'$}$$

$$\underset{\sim}{v}_t - k(1 - \underset{\sim}{v}) - \lambda\phi g(\underset{\sim}{v}) \int_\Omega f(\underset{\sim}{u})\, dx \leqslant 0 \qquad \text{in } (0, T) \tag{2.1d$'$}$$

while the other inequalities remain the same.

Let us consider the following iterative scheme. Given the pairs of "smooth" functions $(\underline{u}_{n-1}, \underline{v}_{n-1})$, $(\bar{u}_{n-1}, \bar{v}_{n-1})$, we define $(\underline{u}_n, \underline{v}_n)$, $(\bar{u}_n, \bar{v}_n)$ as the solution of the following system:

(I) If $\lambda < 0$:

$$\bar{u}_{n,t} - \Delta\bar{u}_n + \phi g(\underline{v}_{n-1}) f(\bar{u}_n) = 0 \qquad \text{in } \Omega \times (0, T) \tag{2.2a}$$

$$\bar{v}_{n,t} - k(1 - \bar{v}_n) - \lambda\phi g(\bar{v}_n) \int_\Omega f(\underline{u}_{n-1})\, dx = 0 \qquad \text{in } (0, T) \tag{2.2b}$$

$$\underline{u}_{n,t} - \Delta\underline{u}_n + \phi g(\bar{v}_{n-1}) f(\underline{u}_n) = 0 \qquad \text{in } \Omega \times (0, T) \tag{2.2c}$$

$$\underline{v}_{n,t} - k(1 - \underline{v}_n) - \lambda\phi g(\underline{v}_n) \int_\Omega f(\bar{u}_{n-1})\, dx = 0 \qquad \text{in } (0, T) \tag{2.2d}$$

$$\bar{u}_n + \alpha\bar{u}_{n,\nu} - 1 = 0 = \underline{u}_n + \alpha\underline{u}_{n,\nu} - 1 \qquad \text{on } \partial\Omega \times (0, T) \tag{2.2e}$$

$$\underline{u}_n(x, 0) = u_0(x) = \bar{u}_n(x, 0) \qquad \text{in } \Omega \tag{2.2f}$$

$$\underline{v}_n(0) = v_0 = \bar{v}_n(0) \tag{2.2g}$$

(II) If $\lambda > 0$, (2.2b) and (2.2d) are replaced by

$$\bar{v}_{n,t} - k(1-\bar{v}_n) - \lambda\phi g(\bar{v}_n) \int_\Omega f(\bar{u}_{n-1})\, dx = 0 \qquad \text{in } (0, T) \qquad (2.2\text{b}')$$

$$\underline{v}_{n,t} - k(1-\underline{v}_n) - \lambda\phi g(\underline{v}_n) \int_\Omega f(\underline{u}_{n-1})\, dx = 0 \qquad \text{in } (0, T). \qquad (2.2\text{d}')$$

This iteration scheme has the effect of decoupling the equations.

If $u_0 \in C^2(\Omega)$ then (2.2) has a unique smooth solution $(\underline{u}_n, \underline{v}_n)$, $(\bar{u}_n, \bar{v}_n)$ (see Amann [1]).

We now prove the following

THEOREM 2.1 (Existence and Comparison). *Let $u_0 \in C^2(\Omega)$ and $(\underset{\sim}{u}, \underset{\sim}{v})$ $(\tilde{u}, \tilde{v})$ pairs of l.u.s. The sequences $(\underline{u}_n, \underline{v}_n)_{n\in N}$, $(\bar{u}_n, \bar{v}_n)_{n\in N}$ obtained from the previous scheme starting from $(\underline{u}_1, \underline{v}_1) = (\underset{\sim}{u}, \underset{\sim}{v})$, $(\bar{u}_1, \bar{v}_1) = (\tilde{u}, \tilde{v})$ converge monotonically to a regular solution (u, v) of (1.1) such that*

$$\underset{\sim}{u} \leqslant u \leqslant \tilde{u} \quad \text{in } \Omega \times [0, \infty) \qquad \underset{\sim}{v} \leqslant v \leqslant \tilde{v} \quad \text{in } [0, \infty).$$

Proof. Let $\lambda < 0$ and $T > 0$. Since

$$\tilde{u}_t - \Delta\tilde{u} + \phi g(\underset{\sim}{v}) f(\tilde{u})$$
$$\geqslant 0 = \bar{u}_{2,t} - \Delta\bar{u}_2 + \phi g(\underset{\sim}{v}) f(\bar{u}_2) \qquad \text{in } \Omega \times (0, T)$$

$$\tilde{v}_t - k(1-\tilde{v}) - \lambda\phi g(\tilde{v}) \int_\Omega f(\underset{\sim}{u})\, dx$$
$$\geqslant 0 = \bar{v}_{2,t} - k(1-\bar{v}_2) - \lambda\phi g(\bar{v}_2) \int_\Omega f(\underset{\sim}{u})\, dx \qquad \text{in } (0, T)$$

$$\underset{\sim}{u}_t - \Delta\underset{\sim}{u} + \phi g(\tilde{v}) f(\underset{\sim}{u})$$
$$\leqslant 0 = \underline{u}_{2,t} - \Delta\underline{u}_2 + \phi g(\tilde{v}) f(\underline{u}_2) \qquad \text{in } \Omega \times (0, T)$$

$$\underset{\sim}{v}_t - k(1-\underset{\sim}{v}) - \lambda\phi g(\underset{\sim}{v}) \int_\Omega f(\tilde{u})\, dx$$
$$\leqslant 0 = \underline{v}_{2,t} - k(1-\underline{v}_2) - \lambda\phi g(\underline{v}_2) \int_\Omega f(\tilde{u})\, dx \qquad \text{in } (0, T)$$

$$\tilde{u} + \alpha\tilde{u}_\nu - 1 \geqslant 0 = \bar{u}_2 + \alpha\bar{u}_{2,\nu} - 1 \qquad \text{on } \partial\Omega \times (0, T)$$
$$\underset{\sim}{u} + \alpha\underset{\sim}{u}_\nu - 1 \leqslant 0 = \underline{u}_2 + \alpha\underline{u}_{2,\nu} - 1 \qquad \text{on } \partial\Omega \times (0, T)$$
$$\underset{\sim}{u}(x, 0) \leqslant \underline{u}_2(x, 0) = u_0(x) = \bar{u}_2(x, 0) \leqslant \tilde{u}(x, 0) \qquad \text{in } \Omega$$
$$\underset{\sim}{v}(0) \leqslant \underline{v}_2(0) = v_0 = \bar{v}_2(0) \leqslant \tilde{v}(0)$$

then by comparison theorems for scalar equations we have

$$\underset{\sim}{u}(x, t) \leqslant \underline{u}_2(x, t), \qquad \bar{u}_2(x, t) \leqslant \tilde{u}(x, t) \qquad \text{in } \Omega \times (0, T)$$

$$\underset{\sim}{v}(t) \leqslant \underline{v}_2(t), \qquad \bar{v}_2(t) \leqslant \tilde{v}(t) \qquad \text{in } (0, T)$$

$g(\underset{\sim}{v}) \leqslant g(\tilde{v})$ implies

$$\bar{u}_{2,t} - \Delta \bar{u}_2 + \phi g(\tilde{v}) f(\bar{u}_2) \geqslant \bar{u}_{2,t} - \Delta \bar{u}_2 + \phi g(\underset{\sim}{v}) f(\bar{u}_2) = 0$$

$$= \underline{u}_{2,t} - \Delta \underline{u}_2 + \phi g(\tilde{v}) f(\underline{u}_2) \qquad \text{in } \Omega \times (0, T)$$

and then

$$\underline{u}_2 \leqslant \bar{u}_2 \qquad \text{in } \Omega \times (0, T).$$

Moreover

$$\bar{v}_{2,t} - k(1 - \bar{v}_2) - \lambda \phi g(\bar{v}_2) \int_\Omega f(\tilde{u})\, dx$$

$$\geqslant \bar{v}_{2,t} - k(1 - \bar{v}_2) - \lambda \phi g(\bar{v}_2) \int_\Omega f(\underset{\sim}{u})\, dx = 0$$

$$= \underline{v}_{2,t} - k(1 - \underline{v}_2) - \lambda \phi g(\underline{v}_2) \int_\Omega f(\tilde{u})\, dx$$

and so

$$\underline{v}_2 \leqslant \bar{v}_2 \qquad \text{in } (0, T).$$

An induction argument shows that for any positive integer n

$$\underline{u}_n(x, t) \leqslant \underline{u}_{n+1}(x, t) \leqslant \bar{u}_{n+1}(x, t) \leqslant \bar{u}_n(x, t) \qquad \text{in } \Omega \times (0, T)$$

$$\underline{v}_n(t) \leqslant \underline{v}_{n+1}(t) \leqslant \bar{v}_{n+1}(t) \leqslant \bar{v}_n(t) \qquad \text{in } (0, T).$$

Using the same argument of Sattinger [7], we can prove that the monotone sequences $(\underline{u}_n, \underline{v}_n)$, $(\bar{u}_n, \bar{v}_n)$ converge to regular solutions $(\underline{u}, \underline{v})$ $(\bar{u}, \bar{v})$ of (1.1).

If $\lambda > 0$ we obtain the same result using the appropriately modified iterative scheme. ∎

THEOREM 2.2 (Uniqueness). *With the assumptions of Theorem* 2.1, *problem* (1.1) *has a unique regular solution which depends continuously on the initial data.*

Proof. Let (u, v) and (u^*, v^*) be two solutions of (1.1) with initial data (u_0, v_0), (u^*, v^*) respectively. It then follows that

$$(u-u^*)_t - \Delta(u-u^*) + \phi g(v)(f(u)-f(u^*)) = \phi f(u^*)(g(v^*)-g(v)).$$

Multiplying both terms by $|\lambda| \operatorname{sign}(u-u^*)$ and integrating over $\Omega \times (0, t)$ we obtain

$$\begin{aligned}
|\lambda| \int_\Omega |u-u^*|\, dx - |\lambda| \int_\Omega |u_0-u_0^*|\, dx + (|\lambda|/\alpha) \int_0^t \left(\int_{\partial\Omega} |u-u^*|\, d\sigma \right) ds& \\
+ |\lambda| \phi \int_0^t g(v) \left(\int_\Omega |f(u)-f(u^*)|\, dx \right) ds& \\
= |\lambda| \phi \int_0^t (g(v^*)-g(v)) \left(\int_\Omega f(u^*) \operatorname{sign}(u-u^*)\, dx \right) ds& \\
\leqslant |\lambda| \phi \int_0^t |g(v^*)-g(v)| \left(\int_\Omega f(u^*)\, dx \right) ds.&
\end{aligned}$$

Moreover

$$\begin{aligned}
(v-v^*)_t = k(v^*-v) + \lambda\phi g(v) \int_\Omega (f(u)-f(u^*))\, dx& \\
+ \lambda\phi(g(v)-g(v^*)) \int_\Omega f(u^*)\, dx.&
\end{aligned}$$

Multiplying both terms by $\operatorname{sign}(v-v^*)$ and integrating over $(0, t)$ we obtain

$$\begin{aligned}
|v-v^*|\,(t) - |v_0-v_0^*| + k \int_0^t |v-v^*|\, ds& \\
= \lambda\phi \int_0^t g(v) \operatorname{sign}(v-v^*) \left(\int_\Omega (f(u)-f(u^*))\, dx \right) ds& \\
+ \lambda\phi \int_0^t |g(v)-g(v^*)| \left(\int_\Omega f(u^*)\, dx \right) ds& \\
\leqslant |\lambda| \phi \int_0^t g(v) \left(\int_\Omega |f(u)-f(u^*)|\, dx \right) ds& \\
+ \lambda\phi \int_0^t |g(v)-g(v^*)| \left(\int_\Omega f(u^*)\, dx \right) ds.&
\end{aligned}$$

Summing the formulas for u and v we have

$$\begin{aligned}
&|\lambda| \int_\Omega |u(x,t)-u^*(x,t)|\,dx + |v-v^*|\,(t)\\
&\quad + (|\lambda|/\alpha)\int_0^t \left(\int_{\partial\Omega} |u-u^*|\,d\sigma\right) ds\\
&\quad - |\lambda| \int_\Omega |u_0(x)-u_0^*(x)|\,dx - |v_0-v_0^*|\\
&\quad + |\lambda|\phi \int_0^t g(v)\left(\int_\Omega |f(u)-f(u^*)|\,dx\right) ds\\
&\leqslant |\lambda|\phi \int_0^t |g(v)-g(v^*)|\left(\int_\Omega f(u^*)\,dx\right) ds\\
&\quad + |\lambda|\phi \int_0^t g(v)\left(\int_\Omega |f(u)-f(u^*)|\,dx\right) ds\\
&\quad + \lambda\phi \int_0^t |g(v)-g(v^*)|\left(\int_\Omega f(u^*)\,dx\right) ds
\end{aligned}$$

and then

$$\begin{aligned}
&|\lambda| \int_\Omega |u(x,t)-u^*(x,t)|\,dx + |v-v^*|\,(t)\\
&\leqslant |\lambda| \int_\Omega |u_0-u_0^*|\,dx + |v_0-v_0^*|\\
&\quad + (\lambda+|\lambda|)\phi \int_0^t |g(v)-g(v^*)|\left(\int_\Omega f(u^*)\,dx\right) ds\\
&\leqslant (\lambda+|\lambda|)M \int_0^t |v-v^*|\,ds,
\end{aligned}$$

where M is proportional to the Lipschitz constant of g. Setting

$$h(t) = |\lambda| \int_\Omega |u(x,t)-u^*(x,t)|\,dx + |v-v^*|\,(t)$$

the previous inequality becomes $h(t) \leqslant h(0) + (\lambda+|\lambda|)M\int_0^t h(s)\,ds$. Then, by Gronwall's lemma, we obtain $h(t) \leqslant h(0)\exp[(\lambda+|\lambda|)Mt]$ and the theorem is proved. Note that if $\lambda<0$ we have the stronger result $h(t) \leqslant h(0)$. ∎

If the initial data is less regular, for instance $u_0 \in L^\infty(\Omega)$, then Theorem 2.1 and 2.2 still hold for weak solutions.

3. The Steady-State Problem

In this section we analyze the stationary system

$$\begin{aligned} \Delta u - \phi g(v) f(u) = 0 &\qquad \text{in } \Omega \\ \alpha\, \partial u/\partial \nu = 1 - u &\qquad \text{on } \partial\Omega \\ k(1-v) + \lambda \phi g(v) \int_\Omega f(u)\, dx = 0. & \end{aligned} \tag{3.1}$$

We can give the definition of pairs of l.u.s. for (3.1) in the same way as in (1.1) dropping the time-dependent terms. So we can prove an existence and comparison theorem in the same way.

Define $\delta^2 = \phi g(v)$. Then the system (3.1) can be seen as the elliptic equation

$$\Delta u - \delta^2 f(u) = 0 \qquad \text{in } \Omega, \qquad \alpha\, \partial u/\partial \nu = 1 - u \qquad \text{on } \partial\Omega, \tag{3.2}$$

where the parameter δ^2 solves the equation

$$\int_\Omega f(u)\, dx = \omega(\delta^2) \tag{3.3}$$

and $\omega(\delta^2) = (k/\lambda)[\ln(\delta^2/\phi)/(\delta^2(\gamma - \ln(\delta^2/\phi)))]$ in $[0, \exp(\gamma)\phi)$.

If δ^2 is fixed, then existence and uniqueness of a regular nonnegative solution of (3.2) is well known (see Friedmann and Phillips [5] and Graham-Eagle and Stakgold [6]).

Now it remains to know how many solutions Eq. (3.2) has.

Theorem 3.1. *If $\lambda > 0$ and $0 \leqslant \gamma \leqslant 4$, then the solution of* (3.1) *is unique.*

Proof. It is known (see Friedmann and Phillips [5]) that the solution $u(x; \delta^2)$ of (3.2) is a nonincreasing function of δ^2. Since ω is an increasing function of δ^2, then Eq. (3.3) has a unique solution in $(\phi, \phi \exp(\gamma))$ and the desired result follows. ∎

If $\lambda > 0$ and $\gamma > 4$ then ω is not monotone and so (3.3) can have more than one solution; that is, we can have multiple steady states. Different and quite involved sufficient conditions for the uniqueness and the global stability of the steady state of the exothermic problem are given in Vega [9] for the case $p \geqslant 1$.

If $\lambda < 0$, then ω is a decreasing function of δ^2 and so we can say nothing about the number of solutions of (3.3). In this case we use a different approach to prove the uniqueness for (3.1).

Theorem 3.2. *If $\lambda < 0$, then* (3.1) *has a unique solution.*

Proof. We prove the uniqueness by contradiction. Assume there exist two solutions (u, v) and (u^*, v^*). If $v = v^*$ then u and u^* solve the same elliptic equation and so $u = u^*$. Suppose $v < v^*$. Then $g(v) < g(v^*)$ and so $u^* < u$ in Ω. This implies that $u_\nu \leqslant u^*_\nu$ on $\partial\Omega$ and so $(k/\lambda)(v-1) = \phi g(v) \int_\Omega f(u)\,dx = \int_\Omega \Delta u\,dx = \int_{\partial\Omega} u_\nu\,d\sigma \leqslant \int_{\partial\Omega} u^*_\nu\,d\sigma = (k/\lambda)(v^*-1)$. It follows that $v^* \leqslant v$ contradicting our assumption. The case $v^* < v$ is similar. ∎

In the rest of this paper we assume that the problem (3.1) has only one solution.

We have the following

LEMMA 3.3. *Let (u, v) the solution of* (3.1) *and v^* the unique positive solution of the equation $k(v^*-1) = \lambda\phi\,|\Omega|\,g(v^*)$ if $\lambda < 0$ and the largest one if $\lambda > 0$. $|\Omega|$ is the measure of Ω.*

(I) *If $\lambda < 0$ then $0 \leqslant u \leqslant 1$ in $\bar{\Omega}$ and $0 < v^* \leqslant v < 1$.*

(II) *If $\lambda > 0$ then $0 \leqslant u \leqslant 1$ in $\bar{\Omega}$ and $1 < v \leqslant v^*$.*

Proof. If $\lambda < 0$, $(0, v^*)$, $(1, 1)$ are pairs of l.u.s. If $\lambda > 0$, $(0, 1)$, $(1, v^*)$ are pairs of l.u.s. ∎

In the rest of this paper we assume that $0 < p < 1$.

In what follows $n \geqslant 2$ is an integer, $\Omega \subset R^n$ is a convex domain with an interior ball property and for each $s \in \partial\Omega$, $a(s)$ is the radius of the largest interior tangent ball at s. Let us define

$$\underset{\sim}{a} = \min_{s \in \partial\Omega} a(s).$$

Graham-Eagle and Stakgold [6] prove the following result about the distance of the dead-core from the boundary.

LEMMA 3.4.

$$\text{(i)}\quad \textit{If } \delta^2 \geqslant k_1/\underset{\sim}{a}^2 \textit{ then } S(H\,\delta^{-2/(1+p)}\alpha^{-(1-p)/(1+p)}) \subset D \tag{3.4}$$

$$\text{(ii)}\quad \textit{If } \delta^2 \geqslant k_2/\alpha^2 \textit{ then } D \subset S(h\,\delta^{-2(1+p)}\alpha^{-(1-p)/(1+p)}), \tag{3.5}$$

where $S(r) = \{x \in \Omega / d(x, \partial\Omega) \geqslant r\}$, $D = \{x \in \Omega / u(x) = 0\}$, u is the solution of (3.2),

$$k_1 = [2/(1-p)][2/(1-p)+n-2],$$
$$k_2 = [(p+1)/2][1/(2^{(p+1)/2}-1)],$$
$$h = [2/(1-p)][(p+1)/2]^{1/(1+p)}\,[1/2]^{(1-p)/2}$$
$$H = [2/(1-p)]^{2p/(1+p)}\,[2/(1-p)+n-2].$$

LEMMA 3.5. *Let $\lambda < 0$. Given $0 < \phi_1 < \phi_2$, let (u_i, v_i) $i = 1, 2$ be the corresponding solutions of* (3.1). *Then $u_2 \leqslant u_1$ in Ω, $v_2 \leqslant v_1$, and*

$$\phi_1 g(v_1) < \phi_2 g(v_2).$$

Proof. If $v_1 = v_2$, then $\phi_1 g(v_1) < \phi_2 g(v_2)$ and by comparison $u_1 \geqslant u_2$ and the lemma is true. Let us assume $v_1 \neq v_2$. If $v_1 < v_2$ then $\phi_1 g(v_1) < \phi_2 g(v_2)$ and $u_2 \leqslant u_1$ in $\bar{\Omega}$. From the boundary conditions we have then $0 \leqslant \partial u_1/\partial v \leqslant \partial u_2/\partial v$ on $\partial\Omega$. This implies that $v_1 = 1 + (\lambda/k) \int_{\partial\Omega} \partial u_1/\partial v \geqslant 1 + (\lambda/k) \int_{\partial\Omega} \partial u_2/\partial v = v_2$, that is $v_2 \leqslant v_1$ which contradicts our assumption. Then it must be $v_2 < v_1$. If $\phi_1 g(v_1) \geqslant \phi_2 g(v_2)$ then $u_1 \leqslant u_2$ and $\partial u_1/\partial v \geqslant \partial u_2/\partial v$. But this implies $v_1 < v_2$ contradicting our previous result. Hence the lemma is proved. ∎

As a consequence of Lemma 3.5 we have that, when $\lambda < 0$, the function $\delta^2(\phi)$ is increasing and then it has a limit as ϕ tends to infinity. Let us denote this limit by L. It is easy to show that v^* defined in Lemma 3.3 tends to zero as ϕ tends to infinity. Then $\lim_{\phi \to \infty} \phi g(v^*) = \lim_{\phi \to \infty} (k/|\lambda|\ |\Omega|)(1 - v^*) = k/|\lambda|\ |\Omega|$ and, since $v \geqslant v^*$, it follows that

$$L \geqslant k/|\lambda|\ |\Omega|. \tag{3.6}$$

LEMMA 3.6. *Let w be the solution of the problem*

$$\Delta w = d^2 w^p \quad \text{in } \Omega$$
$$\alpha w_v = 1 - w \quad \text{on } \partial\Omega.$$

If $d^2 \geqslant k_1/\underline{a}$ then

$$w \leqslant E(\alpha d)^{-2/(1+p)} \quad \text{in } \bar{\Omega},$$

where $k_1 = [2/(1-p)][(2/(1-p)) + n - 2]$ and $E = k_1^{1/(1+p)}[(1-p)/2]^{2/(1+p)}$.

Proof. See Graham-Eagle and Stakgold [6]. ∎

THEOREM 3.7. *Let k_1 and E be as in the previous lemma. Let $\lambda < 0$ and $T = 1 - (\alpha k/|\lambda| S_\Omega)$, where S_Ω is the measure of $\partial\Omega$. Let us assume that $T > 0$. Then*

$$L \leqslant F, \tag{3.7}$$

where $F = \max((k_1/\underline{a}^2), C)$ and $C = (E/T)^{1+p}/\alpha^2$.

Proof. We know that δ^2 is an increasing function of ϕ. If $\delta^2 \leqslant k_1/\underline{a}^2$ for every ϕ then (3.7) is true. If not, a value $\bar{\phi}$ exists such that $\delta^2 \geqslant k_1/\underline{a}^2$

for $\phi \geqslant \bar{\phi}$. Let $\phi \geqslant \bar{\phi}$ and $\phi \geqslant C$. Now we construct a pair of l.u.s. for (3.1) setting $\underset{\sim}{v}=0$, $\tilde{u}=1$, $\tilde{v}=1/[1+\ln(\phi/C)/\gamma]$ and $\underset{\sim}{u}$ the solution of the problem

$$\begin{aligned} \Delta \underset{\sim}{u} &= Cf(\underset{\sim}{u}) && \text{in } \Omega \\ \alpha \underset{\sim}{u}_\nu &= 1-\underset{\sim}{u} && \text{on } \partial\Omega. \end{aligned}$$

It is easy to verify that $\phi g(\tilde{v})=C$. Then $-\Delta \underset{\sim}{u}+\phi g(\tilde{v}) f(\underset{\sim}{u})= -\Delta \underset{\sim}{u}+Cf(\underset{\sim}{u})=0$. Now we prove that

$$-k(1-\tilde{v})-\lambda\phi g(\tilde{v}) \int_\Omega f(\underset{\sim}{u})\, dx \geqslant 0.$$

In fact, by Green's formula and Lemma 3.6 we have

$$\begin{aligned} &-k(1-\tilde{v})-\lambda\phi g(\tilde{v}) \int_\Omega f(\underset{\sim}{u})\, dx \\ &\quad\geqslant -k-\lambda C\int_\Omega f(\underset{\sim}{u})\, dx = -k-\lambda\int_{\partial\Omega} \underset{\sim}{u}_\nu\, ds \\ &\quad= -k-(\lambda/\alpha)\int_{\partial\Omega}(1-\underset{\sim}{u})\, ds = -k-(\lambda/\alpha)\left(S_\Omega-\int_{\partial\Omega}\underset{\sim}{u}\, ds\right) \\ &\quad\geqslant -k-(\lambda/\alpha)\, S_\Omega(1-E/(\alpha^2 C)^{1/(1+p)}) = -k-(\lambda/\alpha)\, S_\Omega(1-T) \\ &\quad= -k+(\lambda/\alpha)\, S_\Omega(\alpha k/\lambda S_\Omega)=0. \end{aligned}$$

It is trivial to verify the other inequalities which occur in the definition of l.u.s. Then, if $\phi \geqslant C$ we have $v \leqslant \tilde{v}$ and so $\delta^2 \leqslant \phi g(\tilde{v})=C$. If $\phi < C$ then the same is true because $v \leqslant 1$. ∎

If $\lambda > 0$ then $v \geqslant 1$ and hence $\lim_{\phi\to\infty} \delta^2(\phi)=\lim_{\phi\to\infty} \phi g(v)=\infty$. Then a value ϕ_1 exists such that $\delta^2 \geqslant \max(k_1/\underset{\sim}{a}^2, k_2/\alpha^2)$ for any $\phi \geqslant \phi_1$ and formulas (3.4), (3.5) hold for the dead-core D of the system (3.1).

If $\lambda < 0$ then the situation can be quite different. From (3.6) we have that if $k/|\lambda|\,|\Omega| > \max(k_1/\underset{\sim}{a}^2, k_2/\alpha^2)$ then a value ϕ_2 exists such that $\delta^2 \geqslant \max(k_1/\underset{\sim}{a}^2, k_2/\alpha^2)$ for any $\phi \geqslant \phi_2$ and again formulas (3.4), (3.5) hold. On the other hand if the assumptions of Theorem 3.7 hold and F is small enough so that $S(hF^{-(1+p)}\alpha^{-(1-p)/(1+p)})$ is empty then D is empty because of (3.5). In this case the solution u of (3.1) is strictly positive in the whole domain Ω and no dead-core occurs.

If Ω is a one-dimensional slab or a ball we could give some further information about the dead-core. Unfortunately this information cannot be used to obtain estimates on the dead-core in the case of general domains. This happens because a comparison theorem between domains is not available in our case; that is, if $\Omega_1 \subset \Omega_2$ nothing can be said about the relation between the corresponding solutions u_1 and u_2.

4. The Time-Dependent Problem

In this section we denote by (u, v) the solution of the problem (1.1). First of all we give bounds for (u, v).

THEOREM 4.1. *Define* $\rho = \max(1, \|u_0\|)$, *where* $\| \ \|$ *is the uniform norm*, $\theta = \max(0, 1 + \lambda\phi\, |\Omega| \exp(\gamma) f(\rho)/k)$. *Then*

$$0 \leqslant u \leqslant \rho \qquad \text{in } \bar{\Omega} \times [0, \infty)$$

and

$$\theta(1 - \exp(-kt)) \leqslant v(t) \leqslant 1 - (1 - v_0)\exp(-kt) \qquad \text{in } [0, \infty) \quad \text{if} \quad \lambda < 0$$

$$1 - (1 - v_0)\exp(-kt) \leqslant v(t) \leqslant v_0 \exp(-kt) + \theta(1 - \exp(-kt)) \qquad \text{in } [0, \infty) \quad \text{if} \quad \lambda > 0.$$

Proof. If $\lambda < 0$ then $(0, \theta(1 - \exp(-kt)))$, $(\rho, 1 - (1 - v_0)\exp(-kt))$ are pairs of l.u.s. If $\lambda > 0$ then $(0, 1 - (1 - v_0)\exp(-kt))$, $(\rho, v_0 \exp(-kt) + \theta(1 - \exp(-kt)))$ are pairs of l.u.s. ∎

Let us note that Theorem 4.1 implies that (u, v) is nonnegative.
Now we compare (u, v) with the solutions of some steady-state problems.

THEOREM 4.2. *Let* ρ *be as in Theorem* 4.1.

(I) *Let* $\lambda < 0$ *and* (u_∞, v_∞) *the solution of the steady-state problem*

$$\begin{gathered} \Delta u_\infty - \phi g(v_\infty) f(u_\infty) = 0 \qquad \text{in } \Omega, \qquad \alpha\, \partial u_\infty/\partial \nu = 1 - u_\infty \qquad \text{on } \partial\Omega \\ k(1 - v_\infty) + \lambda\phi g(v_\infty) \int_\Omega f(u_\infty)\, dx = 0 \end{gathered} \tag{4.1}$$

(i) *If* $u_0 \leqslant u_\infty$ *in* Ω *and* $v_\infty \leqslant v_0 \leqslant 1$ *then* $0 \leqslant u(x, t) \leqslant u_\infty(x)$ *in* $\Omega \times [0, \infty)$, $v_\infty \leqslant v(t) \leqslant 1$ *in* $[0, \infty)$.

(ii) *If* $u_\infty \leqslant u_0$ *in* Ω *and* $v_0 \leqslant v_\infty$ *then* $u_\infty(x) \leqslant u(x, t) \leqslant \rho$ *in* $\Omega \times [0, \infty)$, $0 \leqslant v(t) \leqslant v_\infty$ *in* $[0, \infty)$.

(II) *Let* $\lambda > 0$ *and*

$$v_1 = \max(v_0, 1 + \lambda\phi\, |\Omega| \exp(\gamma)/k)$$

$$v_2 = \max(v_0, 1 + \lambda\phi\, |\Omega| \exp(\gamma) f(\rho)/k).$$

Let u_1 and u_2 be the solutions respectively of the following problems

$$\Delta u_1 = \phi f(u_1) \qquad \text{in } \Omega \qquad \alpha u_{1,\nu} = 1 - u_1 \qquad \text{on } \partial\Omega \tag{4.2}$$

$$\Delta u_2 = \phi \exp(\gamma) f(u_2) \qquad \text{in } \Omega \qquad \alpha u_{2,\nu} = 1 - u_2 \qquad \text{on } \partial\Omega \tag{4.3}$$

(i) *If* $u_0 \leqslant u_1$ *in* Ω *and* $1 \leqslant v_0 \leqslant v_1$, *then* $0 \leqslant u(x,t) \leqslant u_1(x)$ *in* $\Omega \times [0, \infty)$, $1 \leqslant v(t) \leqslant v_1$ *in* $[0, \infty)$.

(ii) *If* $u_2 \leqslant u_0$ *in* Ω *and* $v_0 \leqslant v_2$, *then* $u_2(x) \leqslant u(x,t) \leqslant \rho$ *in* $\Omega \times [0, \infty)$, $0 \leqslant v(t) \leqslant v_2$ *in* $[0, \infty)$.

Proof. The existence and uniqueness of the solution of (4.1) follows from Theorem 3.2 while for (4.2), (4.3) it is well known. If $\lambda < 0$, $u_0 \leqslant u_\infty$ in Ω and $v_\infty \leqslant v_0 \leqslant 1$, then it is easy to show that $(0, v_\infty)$ $(u_\infty, 1)$ are pairs of l.u.s. for (1.1). If $\lambda < 0$, $u_\infty \leqslant u_0$ in Ω and $v_0 \leqslant v_\infty$ then $(u_\infty, 0)$, (ρ, v_∞) are pairs of l.u.s. for (1.1). If $\lambda > 0$, $u_0 \leqslant u_1$ in Ω and $1 \leqslant v_0 \leqslant v_1$, then $(0, 1)$, (u_1, v_1) are pairs of l.u.s. for (1.1). If $\lambda > 0$, $u_2 \leqslant u_0$ in Ω and $v_0 \leqslant v_2$, then $(u_2, 0)$, (ρ, v_2) are pairs of l.u.s. for (1.1). ∎

Let us indicate by $D(t)$, D_∞, D_1, D_2 the dead-cores of the problems (1.1), (4.1), (4.2), (4.3), respectively. Of course $D_2 \supset D_1$ but other inclusions are less obvious. From the previous lemma it follows that:

- If $\lambda < 0$ and $u_0 \leqslant u_\infty$ in Ω, $v_\infty \leqslant v_0 \leqslant 1$ then $D(t) \supset D_\infty$;
- if $\lambda < 0$ and $u_\infty \leqslant u_0$ in Ω, $v_0 \leqslant v_\infty$ then $D(t) \subset D_\infty$;
- if $\lambda > 0$ and $u_0 \leqslant u_1$ in Ω and $1 \leqslant v_0 \leqslant v_1$, then $D(t) \supset D_1$;
- if $\lambda > 0$ and $u_2 \leqslant u_0$ in Ω and $v_0 \leqslant v_2$, then $D(t) \subset D_2$.

In the next two lemmas we compare the solution of (1.1) with the initial data.

LEMMA 4.3. *Let* $\lambda < 0$.

(i) *If* $u_0(x) \leqslant 1$ *and*

$$-k(1 - v_0) - \lambda\phi g(v_0) \int_\Omega f(u_0)\, dx \geqslant 0$$

$$-\Delta u_0 + \phi g(v_0) f(u_0) \leqslant 0 \qquad \text{in } \Omega, \qquad u_0 + \alpha u_{0,\nu} - 1 \leqslant 0 \qquad \text{on } \partial\Omega$$

then

$$u_0(x) \leqslant u(x,t) \leqslant 1, \qquad 0 \leqslant v(t) \leqslant v_0 \qquad \forall (x,t) \in \Omega \times (0, \infty).$$

(ii) *If* $v_0 < 1$ *and*

$$-k(1-v_0) - \lambda\phi g(v_0)\int_\Omega f(u_0)\,dx \leqslant 0$$

$$-\Delta u_0 + \phi g(v_0)\,f(u_0) \geqslant 0 \qquad in\ \Omega, \qquad u_0 + \alpha u_{0,\nu} - 1 \geqslant 0 \qquad on\ \partial\Omega$$

then

$$0 \leqslant u(x,t) \leqslant u_0(x), \qquad v_0 \leqslant v(t) \leqslant 1 \qquad \forall (x,t) \in \Omega \times (0,\infty).$$

Proof. (i) $(u_0, 0)$, $(1, v_0)$ are pairs of l.u.s.; (ii) $(0, v_0)$, $(u_0, 1)$ are pairs of l.u.s. ∎

LEMMA 4.4. *Let* $\lambda > 0$.

(i) *If* $1 \leqslant v_0$, $u_0(x) \leqslant 1$ *and*

$$-k(1-v_0) - \lambda\phi g(v_0)\,|\Omega| \geqslant 0$$

$$-\Delta u_0 + \phi g(v_0)\,f(u_0) \leqslant 0 \qquad in\ \Omega, \qquad u_0 + \alpha u_{0,\nu} - 1 \leqslant 0 \qquad on\ \partial\Omega$$

then

$$u_0(x) \leqslant u(x,t) \leqslant 1, \qquad 1 \leqslant v(t) \leqslant v_0 \qquad \forall (x,t) \in \Omega \times (0,\infty).$$

(ii) *If* $v_0 \leqslant 1$ *and*

$$-k(1-A) - \lambda\phi g(A)\int_\Omega f(u_0)\,dx \geqslant 0$$

$$-\Delta u_0 + \phi g(v_0)\,f(u_0) \geqslant 0 \qquad in\ \Omega, \qquad u_0 + \alpha u_{0,\nu} - 1 \geqslant 0 \qquad on\ \partial\Omega$$

then

$$0 \leqslant u(x,t) \leqslant u_0(x), \qquad v_0 \leqslant v(t) \leqslant A \qquad \forall (x,t) \in \Omega \times (0,\infty).$$

Proof. (i) $(u_0, 1)$, $(1, v_0)$ are pairs of l.u.s.; (ii) $(0, v_0)$, (u_0, A) are pairs of l.u.s. Note that such a constant $A > 1$ exists because $(x-1)/g(x)$ tends to infinity as x tends to infinity. ∎

In the next theorem we give information about the asymptotic behavior of (u, v) for the endothermic case.

THEOREM 4.5. *Let* $\lambda < 0$.

(i) *If the assumptions* (i) *of Lemma* (4.3) *hold, then* $u(x,\cdot)$ *is nondecreasing and* v *is nonincreasing. Moreover* (u, v) *converge to the solution of the steady-state problem as* $t \to \infty$.

(ii) *If the assumptions* (ii) *of Lemma* (4.3) *hold, then* $u(x, \cdot)$ *is nonincreasing and* v *is nondecreasing. Moreover* (u, v) *converge to the solution of the steady-state problem as* $t \to \infty$.

Proof. Let $t \geq 0$, $h > 0$, and define $u^*(x, t) = u(x, t+h)$, $v^*(t) = v(t+h)$.

(i) (u_0, v^*), (u^*, v_0) are pairs of l.u.s. In fact all the inequalities required from the definition of l.u.s. are satisfied. In particular the inequalities on the initial and boundary data are true because of Lemma 4.3. The latter part of the theorem is a straightforward application of the techniques of Sattinger [7].

(ii) (u^*, v_0), (u_0, v^*) are pairs of l.u.s. because of Lemma 4.3. The rest of the theorem follows as in case (i). ∎

If $\lambda > 0$ then Lemma 4.4 cannot be used to obtain asymptotic information about the solution of (1.1) and therefore we cannot exclude oscillations.

THEOREM 4.6. *Let* $\lambda < 0$ *and* $h > 0$.

(i) *If the assumptions of Lemma* 4.3(i) *hold then we have*

$$u(x, t) < u(x, t+h) \qquad D(t+h) \subset D(t)$$
$$d(\partial D(t), D(t+h)) > 0$$

for any $t > 0$ *and* $x \in \Omega \backslash D(t)$.

(ii) *If the assumptions of Lemma* 4.3(ii) *hold then we have*

$$u(x, t) > u(x, t+h) \qquad D(t+h) \supset D(t)$$
$$d(\partial D(t+h), D(t)) > 0$$

for any $t > 0$ *and* $x \in \Omega \backslash D(t+h)$.

In both cases we have also that the dead-core of the steady-state problem cannot be reached in finite time.

Proof. This theorem can be proved following in a straightforward way the proof of Lemma 2.3 in Bandle and Stakgold [3]. ∎

Let (u^*, v^*) be the solution of the ordinary differential system

$$\begin{aligned} du^*/dt &= -\phi g(v^*) f(u^*) \\ dv^*/dt &= k(1 - v^*) + \lambda\phi\, |\Omega|\, g(v^*) f(u^*) \\ u^*(0) &= u_0^* > 0 \qquad v^*(0) = v_0^* \geq 0. \end{aligned} \tag{4.4}$$

The following lemma holds

LEMMA 4.7.

$$\int_0^\infty g(v^*(s))\,ds = +\infty. \tag{4.5}$$

Proof. Let $\lambda < 0$. If a time T exists such that $u^*(t)$ is a constant for any $t \geqslant T$ then we must have $u^* = 0$ and $v^* = 1$ for $t \geqslant T$ and (4.5) follows. Otherwise a time T_1 exists such that v is increasing or decreasing at T_1. If v^* is increasing at T_1 then it must be increasing at any $t > T_1$. If not, a time $T_2 > T_1$ exists such that v^* has a relative maximum at T_2. This implies that, in a right neighborhood of T_2, $v^*(t)$ and dv^*/dt are decreasing. But since u^* is decreasing and $\lambda < 0$ we have that $dv^*/dt = k(1 - v^*) + \lambda\phi\,|\Omega|\,g(v^*)\,f(u^*)$ must be increasing and this is a contradiction. If v^* is decreasing at $t = T_1$ then we must distinguish two cases: (a) v^* has a relative minimum at a time $T_3 > T_1$. Then we must have $v^*(T_3) > 0$, for if we had $v^*(T_3) = 0$ then $dv^*(T_3)/dt = k > 0$ which is absurd. From the previous point it follows that $v^*(t) \geqslant v(T_3) > 0$ for any $t \geqslant T_3$. (b) $v^*(t)$ is decreasing for any $t \geqslant T_1$. Then v^* has a positive limit as t tends to infinity. Let us assume by contradiction that this limit is zero. Choose a positive number ε such that $\varepsilon < 1$ and $\varepsilon/(1 - \varepsilon) < k$. Then a time T_4 exists such that $v(T_4) < \varepsilon$ and $g(v(T_4)) < -\varepsilon/(\lambda\phi\,|\Omega|\,f(u_0^*))$. Therefore

$$\begin{aligned} dv^*(T_4)/dt &\geqslant k(1 - \varepsilon) + \lambda\phi\,|\Omega|\,f(u_0^*)(-\varepsilon/(\lambda\phi\,|\Omega|\,f(u_0^*))) \\ &= k(1 - \varepsilon) - \varepsilon > 0. \end{aligned}$$

Then v^* must be increasing at T_4 which is a contradiction. Therefore we have that in any case a time T^* exists such that for any $t \geqslant T^*$, $v^*(t) > \delta > 0$ and then $g(v^*(t)) > g(\delta) > 0$. Formula (4.5) immediately follows. If $\lambda > 0$ then v^* is increasing at any time t such that $v^*(t) < 1$ and then the assertion easily follows. ∎

From Lemma 4.7 we have the following

LEMMA 4.8. *A time T exists such that $u^*(t) = 0$ for any $t \geqslant T$.*

Proof. From (4.4) we have

$$u^*(t) = [(u_0^*)^{1-p} - (1 - p)\phi \int_0^t g(v(s))\,ds]_+^{1/(1-p)} \tag{4.6}$$

and then the desired result immediately follows from (4.5). ∎

If $\lambda < 0$ we can use the previous result to obtain a comparison between the solutions of (1.1) and (4.4).

THEOREM 4.9. *If* $\lambda < 0$ *and* $0 < u_0(x) \leqslant 1$ *for any* $x \in \bar{\Omega}$ *and* $u_0^* = \min_{x \in \bar{\Omega}} u_0(x) > 0$, $v_0^* = v_0$ *then*

$$u^*(t) \leqslant u(x, t) \leqslant 1, \qquad 0 \leqslant v(t) \leqslant v^*(t) \qquad \text{for} \quad (x, t) \in \Omega \times (0, \infty).$$

Proof. $(u^*, 0)$, $(1, v^*)$ are pairs of l.u.s. for (1.1). ∎

If T is as defined in Lemma 4.8, then from the previous theorem we have that $u(x, t) > 0$, for any $t < T$; that is, the dead-core $D(t)$ is empty before T.

If $\lambda > 0$ no comparison seems to be possible between the solutions of (1.1) and (4.4).

We conclude with a lower bound for the time $\tau = \inf\{t : D(t) \neq \varnothing\}$.

THEOREM 4.10. *If the assumptions of Theorem* 4.9 *hold and* $v_0 \leqslant 1$ *then*

$$\tau + \exp(-k\tau)/k \geqslant 1/k + \exp(-\gamma)(u_0^*)^{(1-p)}/[(1-p)\phi].$$

Proof. Theorem 4.9 implies that $\tau \geqslant T$. From (4.6) we have that T is the solution of the equation

$$\int_0^T g(v^*(s))\, ds = (u_0^*)^{(1-p)}/[(1-p)\phi]. \tag{4.7}$$

Equation (4.4) implies that $dv^*/dt \leqslant k(1 - v^*)$ and therefore $v^*(t) \leqslant [1 - \exp(-kt)]$. Let us remark that $g(v) \leqslant \exp(\gamma) v$ for any $v \in [0, 1]$. Then

$$\begin{aligned} \int_0^T g(v^*(s))\, ds &\leqslant \int_0^T g([1 - \exp(-ks)])\, ds \\ &\leqslant \exp(\gamma) \int_0^T [1 - \exp(-ks)]\, ds \\ &= \exp(\gamma)[T + (1/k)\exp(-kT) - (1/k)]. \end{aligned}$$

From (4.7) we have $\exp(\gamma)[T + (1/k)\exp(-kT) - (1/k)] \geqslant (u_0^*)^{(1-p)}/[(1-p)\phi]$ and, since $h(t) = t + \exp(-kt)/k$ is increasing, the assertion follows. ∎

ACKNOWLEDGMENTS

It is a pleasure to acknowledge useful discussions with Prof. I. Stakgold.

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